Recognizable morphisms and a decision algorithm for substitutive languages.

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- Recognizability

- Decidability property algorithm

In collaboration with F. Dolce, J. Leroy

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Basic Concepts

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$A = \{0, 1\}$

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$$\varphi_F^2(0) = 010 \qquad \varphi_F^2(1) = 01 \qquad \Rightarrow \qquad \varphi_F \ \textbf{primitive}$$

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An infinite word $\mathbf{z}$ is a fixed point of a morphism φ if $\varphi(\mathbf{z}) = \mathbf{z}$.

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$$\begin{aligned} \varphi_F : 0 &\mapsto 01, \\ 1 &\mapsto 0. \end{aligned}$$

The Fibonacci word:

$\varphi_F(0100101001001010010100100\ldots) = 0100101001001010010100100\ldots$

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- The language of a fixed point $\mathbf{z}$ is the set of all its factors and we symbolize it as $\mathcal{L}(\mathbf{z})$. This language is called substitutive.

Example (Fibonacci word)

The substitutive language:

$\mathcal{L}(\mathbf{z}) = \{\varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, 0010, 0100, \ldots\}$

$A^{\mathbb{N}}$: Let $a \in A$ with $\varphi(a) \in aA^*$ and $|\varphi^n(a)|$ tends to infinity with n. Then there is a unique right infinite word $\mathbf{x} = \varphi^\omega(a)$ that satisfies $\varphi(\mathbf{x}) = \mathbf{x}$.

Example

$A = \{0, 1, 2\}$

$$\begin{aligned} \varphi : A^* &\rightarrow A^*, \\ 0 &\mapsto 02, \\ 1 &\mapsto 201, \\ 2 &\mapsto 212. \end{aligned}$$

$$\mathcal{L}(\varphi) = \{\varepsilon, 0, 1, 2, 01, 02, 12, 20, 21, 22, \dots\}$$

Right infinite fixed point:
$\varphi^\omega(0) = 022122122\dots$
$\varphi^\omega(2) = 212201212\dots$

$A^{-\mathbb{N}}$ Let $b \in A$ such that $\varphi(b) \in A^* b$ and $|\varphi^n(b)|$ tends to infinity with n. Then there is a unique left infinite word $\mathbf{y} = \varphi^{\tilde{\omega}}(b)$ that satisfies $\varphi(\mathbf{y}) = \mathbf{y}$.

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$A^{\mathbb{Z}}$ The bi-infinite word $\mathbf{z} = \varphi^{\tilde{\omega}}(b) \bullet \varphi^{\omega}(a)$ has all words $\varphi^n(a)$ and all words $\varphi^n(b)$ as factors, for all n, and satisfies $\varphi(\mathbf{z}) = \mathbf{z}$.

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- A two-sided fixed point $\mathbf{z}$ of a morphism φ is called *admissible* if $\mathbf{z}_{-1}\mathbf{z}_0$ is in the language $\mathcal{L}(\varphi)$

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Two-sided infinite fixed point:

$\varphi^{\tilde{\omega}}(1) \bullet \varphi^{\omega}(0) = \dots 21202201 \bullet 022122122\dots$ $\quad 10 \notin \mathcal{L}(\varphi) \Rightarrow$ **not** admissible
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$\varphi^{\tilde{\omega}}(2) \bullet \varphi^{\omega}(0) = \dots 12201212 \bullet 022122122\dots$ $\quad 20 \in \mathcal{L}(\varphi) \Rightarrow$ admissible
$\varphi^{\tilde{\omega}}(2) \bullet \varphi^{\omega}(2) = \dots 12201212 \bullet 212201212\dots$ $\quad 22 \in \mathcal{L}(\varphi) \Rightarrow$ admissible

Definition

A right infinite word $\mathbf{z}$ is *ultimately periodic* if there are integers $p > 0$, $k_0 \geq 0$ such that for all $k, n \in \mathbb{N}$, with $0 \leq n < p$ and $k \geq k_0$ we have $\mathbf{z}_{n+kp} = \mathbf{z}_n$.

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The right infinite word

$$x = abcbcbcbcbc \cdots = a(bc)^{\omega}$$

is ultimately periodic with $k_0 = 1$ and $p = 2$.

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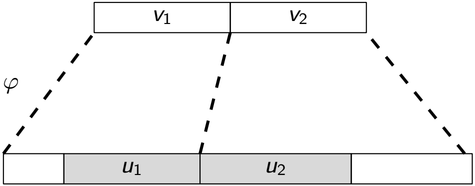
Definition

A primitive morphism with an aperiodic fixed point is called *aperiodic*.

Contents

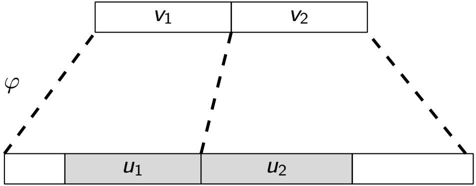
Let $u \in \mathcal{L}(\varphi)$. We say that (u_1, u_2) is a *cutting point* of u for the morphism φ, if:

- $u = u_1 u_2$ and,
- $\exists v_1 v_2 \in \mathcal{L}(\varphi)$ such that:
  - u_1 suffix of $\varphi(v_1)$ and
  - u_2 prefix of $\varphi(v_2)$.



Let $u \in \mathcal{L}(\varphi)$. We say that (u_1, u_2) is a *synchronization point* of u for the morphism φ, if:

- $u = u_1 u_2$ and,
- $\forall v \in \mathcal{L}(\varphi)$ such that u is factor of $\varphi(v) \Rightarrow \exists v_1 v_2$ factorization of v with
  - u_1 suffix of $\varphi(v_1)$ and
  - u_2 prefix of $\varphi(v_2)$.



Any synchronization point is a cutting point.

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$

Example (Fibonacci)

Let φ_F be the Fibonacci morphism with fixed point,

$$\mathbf{z} = \varphi_F^\omega(0) = 01\mathbf{00101}001\mathbf{00101}001010010\ldots.$$

0	0	1	0	1

The word $00101 \in \mathcal{L}(\mathbf{z})$ has four synchronization points $(\varepsilon, 00101)$, $(0, 0101)$, $(001, 01)$ and $(00101, \varepsilon)$, i.e., $|0|01|01|$.

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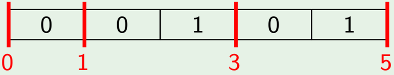

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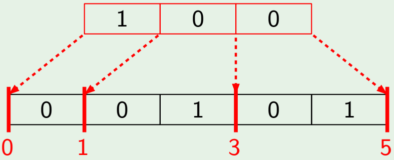

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Let $\mathbf{z}$ be a fixed point of a morphism φ.
We will define the set $E_{\mathbf{z}}$ as follows,

$$E_{\mathbf{z}} = \{0\} \cup \{|\varphi(\mathbf{z}_{[0,p-1]})|; p > 0\}.$$

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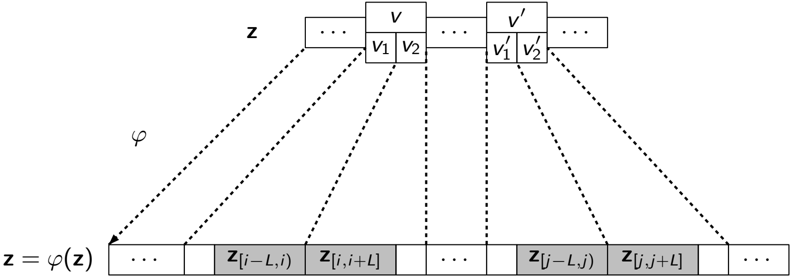
Recognizability [Mossé (1992)]

Let $\varphi : A^* \to A^*$ be a morphism with a fixed point $\mathbf{z}$. We say that φ is *recognizable* on $\mathbf{z}$ if there is an integer $L > 0$ such that for every $i > L$ if

$$\mathbf{z}_{[i-L,i+L]} = \mathbf{z}_{[j-L,j+L]} \text{ and } i \in E_{\mathbf{z}} \qquad \Rightarrow \qquad j \in E_{\mathbf{z}}.$$

In other words: We can recognize the elements in $E_{\mathbf{z}}$ by reading through a window of a size $2L + 1$.



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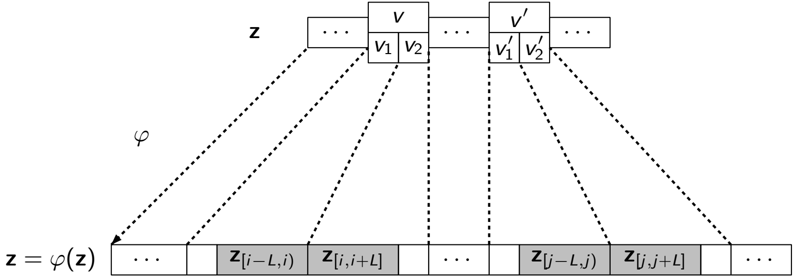

Equivalent form

$\forall u_1 \in \mathcal{L}(\varphi) \cup A^L$, $u_2 \in \mathcal{L}(\varphi) \cup A^{L+1}$ (u_1, u_2) is a C.P. $\Rightarrow$ (u_1, u_2) is a S.P..

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$

Example (Fibonacci)

Let φ_F be the Fibonacci morphism with fixed point,

$$\mathbf{z} = \varphi_F^{\omega}(0) = 0100101001001010010100100 \ldots.$$

For any word of length 3:

- There is a cutting point in the position between two 0's.
- There is a cutting point after each appearance of the letter 1 .

$$\forall u_1 \in \mathcal{L}(\varphi) \cup A^{L},\ u_2 \in \mathcal{L}(\varphi) \cup A^{L+1}\ (u_1, u_2) \text{ is a C.P.} \Rightarrow (u_1, u_2) \text{ is a S.P..}$$

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$0|01|$ $\quad 1|0|0$ $\quad 1|01|$

$0\cancel{|}1|0$

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Example (Fibonacci)

Let φ_F be the Fibonacci morphism with fixed point,

$$\mathbf{z} = \varphi_F^{\omega}(0) = 0100101001001010010100 \ldots .$$

For any word of length 3:

- There is a cutting point in the position between two 0's.
- There is a cutting point after each appearance of the letter 1 .

$0|01|$ $\quad 1|0|0 \quad 1|01|$

$0\,\cancel{|}1|0 \quad \Rightarrow \quad L = 1.$

$$\forall u_1 \in \mathcal{L}(\varphi) \cup A^L,\ u_2 \in \mathcal{L}(\varphi) \cup A^{L+1}\ (u_1, u_2) \text{ is a C.P.} \Rightarrow (u_1, u_2) \text{ is a S.P..}$$

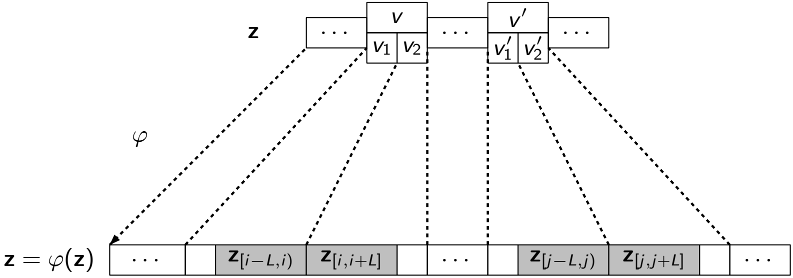
Strong recognizability

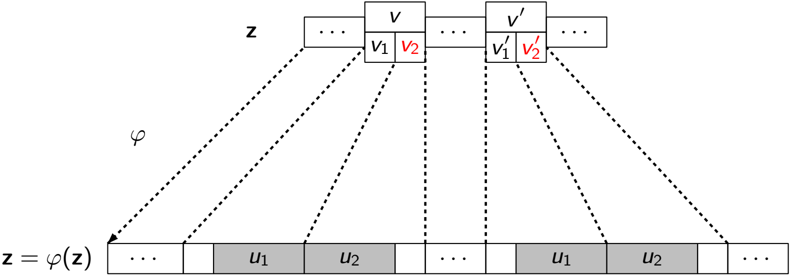
The morphism φ is *strongly recognizable* on $\mathbf{z}$ iff there is an integer $L > 0$ such that for every $i > L$

$$\text{if } \mathbf{z}_{[i-L,i+L]} = \mathbf{z}_{[j-L,j+L]} \text{ and } i \in E_{\mathbf{z}} \quad \Rightarrow \quad j \in E_{\mathbf{z}} \text{ and } \mathbf{z}_{i'} = \mathbf{z}_{j'},$$

for $f(i') = i$ and $f(j') = j$.

Where, $f: \quad i \quad \mapsto \quad f(i) = \begin{cases} |\varphi(\mathbf{z}_{[0,i)})| & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$





Equivalent form

A morphism φ is strongly recognizable if there is integer $L > 0$ such that
$\forall u_1 \in \mathcal{L}(\varphi) \cup A^L, u_2 \in \mathcal{L}(\varphi) \cup A^{L+1}$ where (u_1, u_2) is a C.P. $\Rightarrow$
$\exists a \in A$ s.t. $\forall v \in \mathcal{L}(\varphi)$ where u is factor of $\varphi(v)$, $\exists v_1 v_2$ factorization of v with

- u_1 suffix of $\varphi(v_1)$,
- u_2 prefix of $\varphi(v_2)$ and
- $v_2 \in aA^*$.

$$\varphi_F : \begin{array}{ccc} 0 & \mapsto & 01, \\ 1 & \mapsto & 0. \end{array}$$

Example (Fibonacci)

The morphism φ_F with fixed point the Fibonacci word is strongly recognizable for $L = 1$:

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Example (Fibonacci)

The morphism φ_F with fixed point the Fibonacci word is strongly recognizable for $L = 1$:

$0|01| \Rightarrow (0, 01)$ is S.P. and $001 = \varphi(10)$,

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Example (Fibonacci)

The morphism φ_F with fixed point the Fibonacci word is strongly recognizable for $L = 1$:

$0|01| \Rightarrow (0, 01)$ is S.P. and $001 = \varphi(10)$,

$1|0|0 \Rightarrow (1, 00)$ is S.P. and $0100 = \varphi(01)0$,

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$1|01| \Rightarrow (1, 01)$ is S.P. and $0101 = \varphi(00)$,

$0 \not| 1|0 \Rightarrow (0, 10)$ is **not** C.P. $\Rightarrow$ **not** S.P..

Mossé's recognizability Theorem [1992, 1996]

Every primitive aperiodic morphism is strongly recognizable on any of its right fixed points.

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Two-sided infinite fixed points:

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Two-sided infinite fixed points:

Theorem

Every primitive aperiodic morphism is strongly recognizable on each of its two-sided admissible fixed-points.

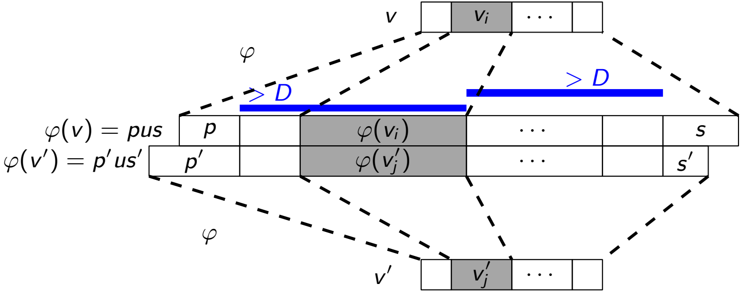
Circularity [Klouda (2012)]

Let φ be a morphism with fixed point $\mathbf{z} \in A^{\mathbb{N}}$. Let $\varphi(v) = pus$ and $\varphi(v') = p'us'$ for non-empty words $u, v, v' \in \mathcal{L}(\mathbf{z})$ with $v = v_0 v_1 \dots v_n$ and $v' = v'_0 v'_1 \dots v'_m$.
We say that φ is *circular* with delay $D > 0$ if whenever we have

$$|\varphi(v_0 \dots v_i)| - |p| > D \text{ and } |\varphi(v_{i+1} \dots v_n)| - |s| > D$$

for some $0 \leq i \leq n$, then there is $0 \leq j \leq m$ such that

$$|\varphi(v_0 \dots v_{i-1})| - |p| = |\varphi(v'_0 \dots v'_{j-1})| - |p'| \qquad \text{and} \qquad v_i = v'_j.$$



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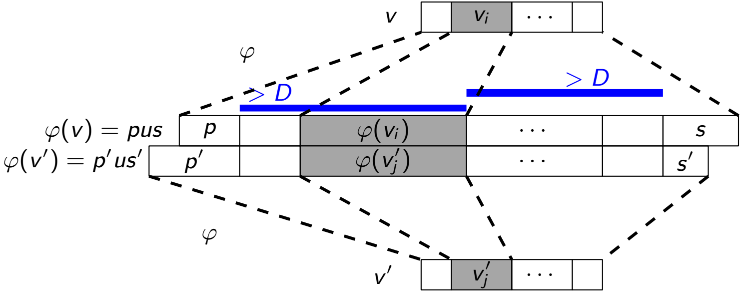

In other words: a long enough word has unique φ-preimage except for some prefix and suffix shorter than a constant $D + 1$.

Result: Equivalence

Theorem [R. Kyriakoglou (2019)]

A morphism is circular if and only if it is strongly recognizable.

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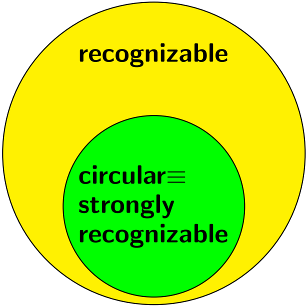

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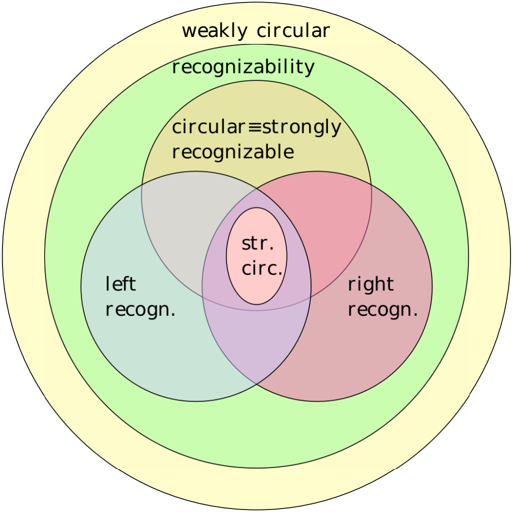


Contents

Antecedent:

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$

Motivation:

Example (Fibonacci)

Possible preimages of $u = 010$:

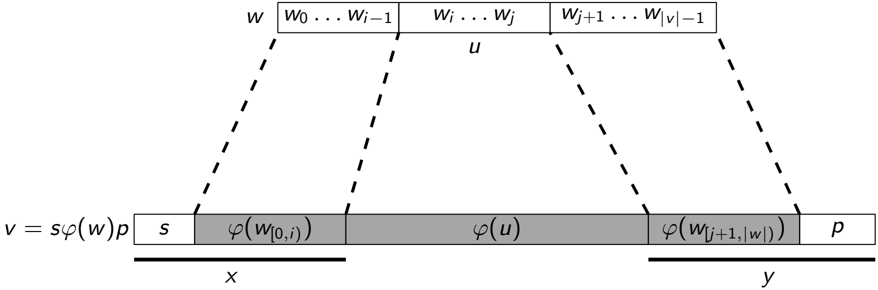
$\varphi_F(01) = 010$ and $\varphi_F(00) = 0101$

Longest common factor of the preimages 01 and 00 : 0

Definition (Antecedent)

Let $\varphi : A^* \to A^*$ be a morphism with fixed point $\mathbf{z}$ and $v \in \mathcal{L}(\mathbf{z})$. The *antecedent* of v in $\mathcal{L}(\mathbf{z})$ (if it exists) is the longest non-empty word $u \in \mathcal{L}(\mathbf{z})$ such that

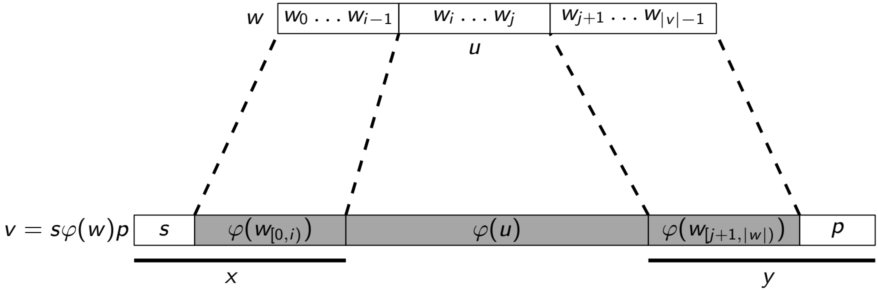
1. $v = x\varphi(u)y$ for some words $x, y \in A^*$ and
2. for any word $w \in \mathcal{L}(\mathbf{z})$ such that $v = s\varphi(w)p$, there exists $i, j \in \{0, \ldots, |w| - 1\}$, $i \leq j$, such that $u = w_{[i,j]}$, $x = s\varphi(w_{[0,i)})$ and $y = \varphi(w_{[j+1,|v|)})p$.



Proposition [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

If φ is a strongly recognizable morphism on a fixed point $\mathbf{z}$ for a constant $L > 0$, then for any word $v \in \mathcal{L}(\mathbf{z})$ of length at least $2L + ||\varphi||$ there is an antecedent u.

Furthermore, if $v = x\varphi(u)y$, then we have $|x| < L + ||\varphi||$ and $|y| < L + 1$.



Corollary

Let φ be a morphism with a fixed point $\mathbf{z}$. If φ is strongly recognizable for some constant $L > 0$ on $\mathbf{z}$, then for all $u \in \mathcal{L}(\mathbf{z})$ with length at least $2L + ||\varphi||$, there exists a unique finite sequence $(u_1, u_2, \ldots, u_k)$ of words in $\mathcal{L}(\mathbf{z})$ such that

1. $|u_1| < 2L + ||\varphi||$;
2. u_i *is the antecedent of* u_{i+1};
3. $u_k = u$.

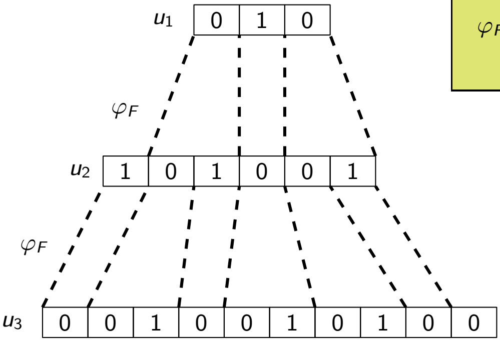
The previous sequence will be called *the sequence of antecedents of u* in $\mathcal{L}(\mathbf{z})$.

Example (Fibonacci)

The Fibonacci morphism φ_F with fixed point the Fibonacci word is strongly recognizable for a constant $L = 1$.

- $u_3 = 0010010100$ has length $|u_3| = 10 \geq 2 \cdot 1 + 2$.
- The antecedent of u_3 is $u_2 = 101001$, with $|u_2| = 6 \geq 2 \cdot 1 + 2$.
- The antecedent of u_2 is $u_1 = 010$, with $|u_1| = 3 < 2 \cdot 1 + 2$.

Thus, for the word u_3 the sequence of antecedents is $(010, 101001, 0010010100)$.



$$\varphi_F : \begin{array}{lcl} 0 & \mapsto & 01, \\ 1 & \mapsto & 0. \end{array}$$

Extension graph

The extension graph of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}^{(n,m)}(w)$ with vertices the disjoint union of $L_S^{(n)}(w)$ and $R_S^{(m)}(w)$ and edges $B_S^{(n,m)}(w)$, where

$$\begin{aligned} L_S^{(n)}(w) &= \{u \in A^n \text{ such that } uw \in S\}, \\ R_S^{(m)}(w) &= \{v \in A^m \text{ such that } wv \in S\}, \\ B_S^{(n,m)}(w) &= \{(u,v) \in A^n \times A^m \text{ such that } uwv \in S\}. \end{aligned}$$

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A word w is called bispecial if $\mathrm{Card}(L_S^{(n)}(w)) \geq 2$ and $\mathrm{Card}(R_S^{(m)}(w)) \geq 2$.

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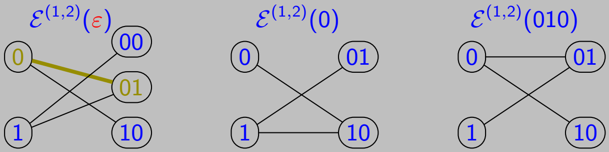
Example (Fibonacci)

$\mathcal{L}(\varphi_F) = \{\varepsilon,\ 0, 1, 00, 01, 10,\ 001, 010, 100,\ 101, \dots\}.$



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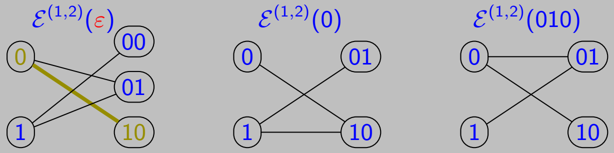

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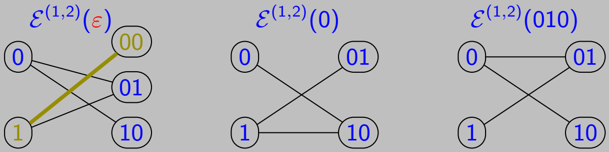

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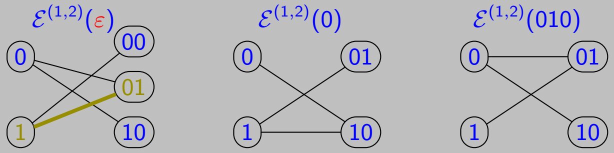

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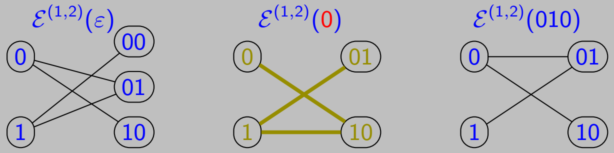

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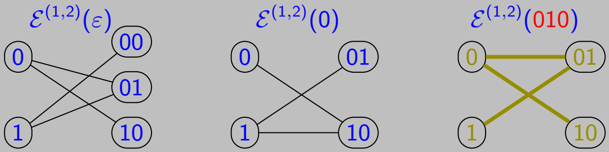

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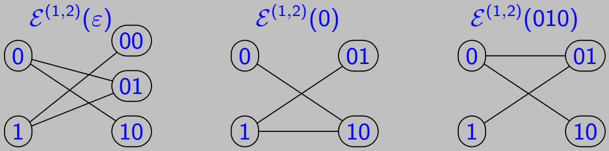
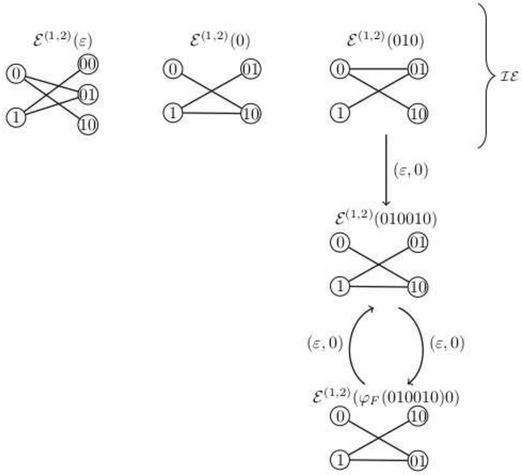


A biextendable set S is called dendric if for any $w \in S$ the graph $\mathcal{E}(w)$ is a tree.

Our goal: Decide if a set is dendric.

Method: construct an algorithm to describe the extension graphs of all bispecial words in a language.



We define an equivalence relation $\mathcal{R}$ on $\mathcal{L}(\mathbf{z}) \times \mathcal{L}(\mathbf{z})$ as follows:
for $u \neq u'$

$$(u, u') \in \mathcal{R} \quad \Longleftrightarrow \quad \begin{cases} |u|, |u'| \geq 2L + ||\varphi|| \\ \mathcal{E}^{(n,m)}(u) = \mathcal{E}^{(n,m)}(u') \end{cases}$$

Let $[u]$ be the corresponding class of $u \in \mathcal{L}(\mathbf{z})$ for the relation $\mathcal{R}$.

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Adequate language

Let us call *adequate* a substitutive language such that for any bispecial words w_1, w_2 with $(w_1, w_2) \in \mathcal{R}$, if $u_1 = x\varphi(w_1)y$ has w_1 as antecedent, then the word w_2 is the antecedent of the word $u_2 = x\varphi(w_2)y$ and $(u_1, u_2) \in \mathcal{R}$.

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We say that a morphism with a fixed point $\mathbf{z}$ is *adequate* if the language $\mathcal{L}(\mathbf{z})$ is adequate.

Proposition [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let φ be a strongly recognizable morphism on an alphabet A. If the morphism φ is stable, then φ is adequate.

A morphism φ is *stable* if there are words $u, v \in A^*$ such that for any two distinct letters $a, b \in A$ we have $f_L(a, b) = u$ and $f_R(a, b) = v$.
Where,

$$f_L(a, b) = \text{longest common suffix of } \varphi(a) \text{ and } \varphi(b), \text{ for } a, b \in A^*$$
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$$f_L(a, b) = \text{longest common suffix of } \varphi(a) \text{ and } \varphi(b), \text{ for } a, b \in A^*$$
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Example (Fibonacci)

The Fibonacci morphism φ_F is stable since it is on the binary alphabet.

Proposition [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let φ be a strongly recognizable morphism on an alphabet A. If the morphism φ is stable, then φ is adequate.

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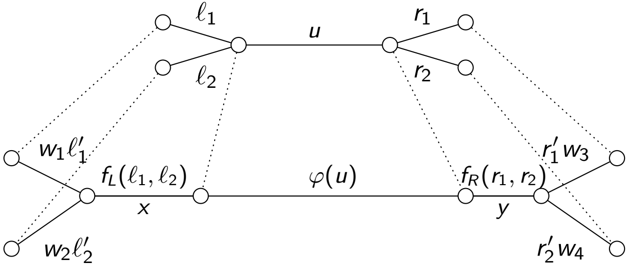
Let φ be a strongly recognizable morphism on an alphabet A. If the morphism φ is bifix, then φ is adequate.

Some important results

Let v be a **bispecial word** with antecedent u, with $v = x\varphi(u)y$.

- The antecedent u of a bispecial word v is bispecial.
- $\mathrm{Card}(B^{(D_L,D_R)}(v)) \leq \mathrm{Card}(B^{(D_L,D_R)}(u))$.
- $B^{(D_L,D_R)}(v) = \left\{ (\ell', r') \mid \exists\, (\ell, r) \in B^{(D_L,D_R)}(u) \text{ s.t. } \begin{array}{l} \varphi(\ell) \in A^*\ell' x \\ \varphi(r) \in y r' A^* \end{array} \right\}.$



Let φ be a strongly recognizable morphism on a fixed point $\mathbf{z}$, for a constant $L > 0$. The set of initial bispecial words is:

$$\mathcal{IE} = \{u | u \in \mathcal{L}(\mathbf{z}), |u| < 2L + ||\varphi||\}$$

and the set of main bispecial words is:

$$\mathcal{BE} = \{u | u \in \mathcal{L}(\mathbf{z}), |u| \geq 2L + ||\varphi||\}.$$

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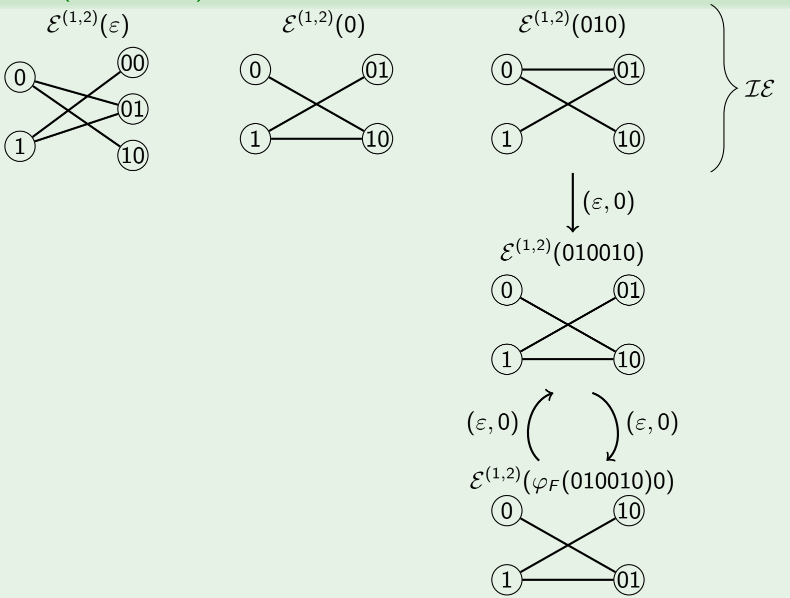
Graph of extension graphs [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let us consider the labeled directed graph $\mathcal{K}(\mathbf{z})$, having set of vertices $\mathcal{E}(\mathcal{IE} \sqcup (\mathcal{BE}/\mathcal{R}))$ and set of edges defined as follows: there is an edge going from G to H, labeled $(f_L(\ell_1, \ell_2), f_R(r_1, r_2))$, if

1. $G = \mathcal{E}^{(D_L, D_R)}(u)$, where $u \in \mathcal{IE} \cup (\mathcal{BE}/\mathcal{R})$,
2. $H = \mathcal{E}^{(D_L, D_R)}(v)$, where $v \in (\mathcal{BE}/\mathcal{R})$,
3. the word u is the antecedent of the word v with $v = f_L(\ell_1, \ell_2)\varphi(u)f_R(r_1, r_2)$ such that $((\ell_1, r_1), (\ell_2, r_2))$ is a bispecial pair in $\mathcal{E}^{(D_L, D_R)}(u)$.

Example (Fibonacci)



Theorem [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let φ be a strongly recognizable morphism on the adequate substitutive language of a fixed point $\mathbf{z}$. The graph $\mathcal{K}(\mathbf{z})$ is finite and computable and if v is a bispecial word in $\mathcal{L}(\mathbf{z})$, then $\mathcal{E}^{(D_L,D_R)}(v) \in \mathcal{K}(\mathbf{z})$.

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- all possible extension graphs of the bispecial words appear in the graph $\mathcal{K}(\mathbf{z})$,
- we construct the graph by increasing length of words,
- we use words shorter than a constant $K = \Phi^{\delta}(2L + ||\varphi|| - 1)$, where $\Phi(n) = n||\varphi|| + C_L + C_R$.

Theorem [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let φ be an adequate strongly recognizable morphism on a fixed point $\mathbf{z}$. Then it is decidable whether the substitutive language $\mathcal{L}(\mathbf{z})$ is a dendric set.

A. Frid and **S. V. Avgustinovich**[1] have proved the same result for bifix morphisms.

[1] Anna E. Frid and Sergey V. Avgustinovich. On bispecial words and subword complexity of D0L sequences. In C. Ding, T. Helleseth, and H. Niederreiter, editors, Sequences and their Applications, pages 191-204. Springer London, 1999.

Conclusion:

Contribution:

- Exploring the different notions of recognizability
- Decidability property

Perspectives:

- Complexity of algorithms
- Check the decidability of larger classes.

Questions?

Details of the sketch

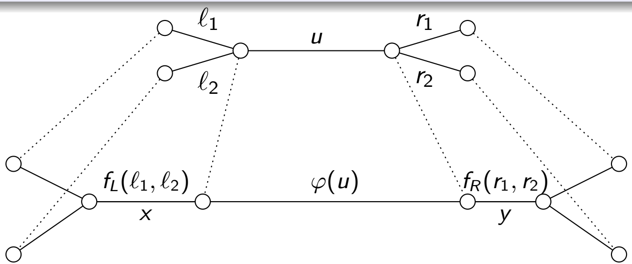
$B_{x,y}^{(m,n)}(u) = \{(\ell, r) \in B^{(m,n)}(u) \mid \varphi(\ell) \in A^* x \text{ and } \varphi(r) \in yA^*\}$

A pair $((\ell_1, r_1), (\ell_2, r_2)) \in B_{x,y}^{(m,n)}(w) \times B_{x,y}^{(m,n)}(w)$ is *bispecial* if:

- ℓ_1, ℓ_2 have distinct last letter and
- r_1, r_2 have distinct first letter.

Proposition [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

If $u \in \mathcal{L}(\mathbf{z})$ is bispecial then there is a bispecial pair in $B_{x,y}^{(m,n)}(u) \times B_{x,y}^{(m,n)}(u)$.

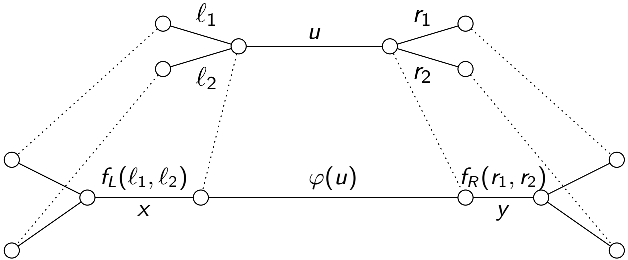


Proposition [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

The antecedent u of a bispecial word v, with $v = x\varphi(u)y$, is bispecial. Furthermore, there is a bispecial pair

$$((\ell_1, r_1), (\ell_2, r_2)) \in B_{x,y}^{(C_L, C_R)}(u) \times B_{x,y}^{(C_L, C_R)}(u)$$

with $x = f_L(\ell_1, \ell_2)$ and $y = f_R(r_1, r_2)$.
In particular, $|y| < C_R$ and $|x| < C_L$, for some computable constants $C_L, C_R \geq 1$.



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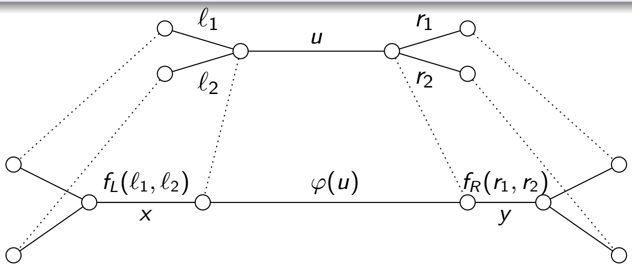

Remark

There exist two computable constants D_L and D_R such that for all $\ell \in A^{D_L}$ and all $r \in A^{D_R}$, $|\varphi(\ell)| \geq D_L + C_L - 1$ and $|\varphi(r)| \geq D_R + C_R - 1$.

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Example (Fibonacci)

Let φ_F be the Fibonacci morphism with fixed point $\mathbf{z}$ the Fibonacci word,

$$C_L = 1 \text{ and } C_R = 2.$$

Also,

$$D_L = 1 \text{ and } D_R = 2.$$

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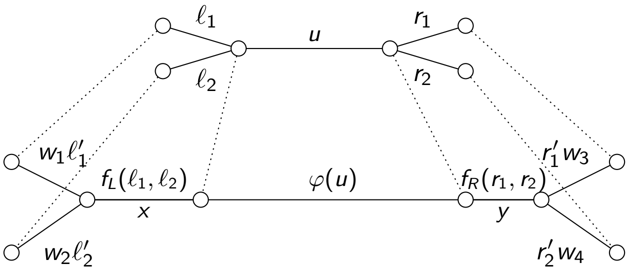
Extensions of a word and extensions of the antecedent

Lemma [F. Dolce, R. Kyriakoglou, J. Leroy (2020)]

Let $v \in \mathcal{L}(\mathbf{z})$ bispecial with antecedent $u \in \mathcal{L}(\mathbf{z})$ such that $v = x\varphi(u)y$. Then,

$$\mathrm{Card}(B^{(D_L,D_R)}(v)) \leq \mathrm{Card}(B^{(D_L,D_R)}_{x,y}(u)).$$



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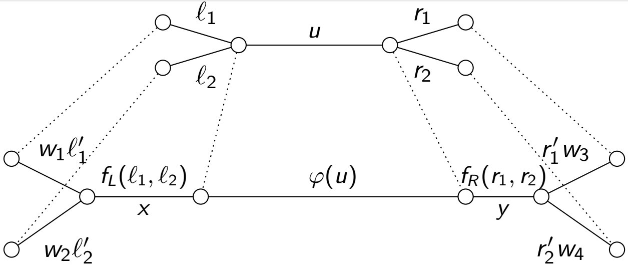


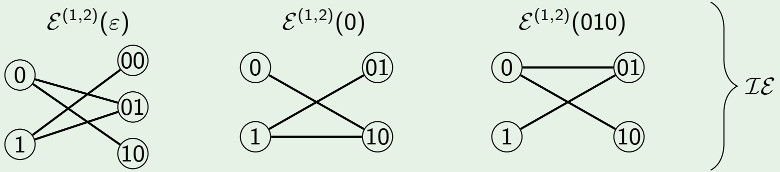
Corollary

$$B^{(D_L,D_R)}(v) = \left\{ (\ell', r') \mid \exists\, (\ell, r) \in B^{(D_L,D_R)}_{x,y}(u) : \begin{array}{l} \varphi(\ell) \in A^*\ell' x \\ \varphi(r) \in y r' A^* \end{array} \right\}.$$

$$\varphi_F : \begin{array}{rcl} 0 & \mapsto & 01, \\ 1 & \mapsto & 0. \end{array}$$

Example (Fibonacci)

$\mathcal{IE}$**:** Bispecial words with length less than 4, since $4 = 2L + ||\varphi|| = 4$.



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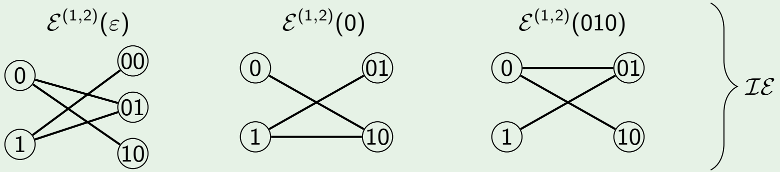

$\mathcal{BE}$:

- ε: $\forall$ bispecial pair $((\ell_1, r_1), (\ell_2, r_2)) \in B^{(1,2)}(\varepsilon) \times B^{(1,2)}(\varepsilon)$,

$$f_L(\ell_1, \ell_2) = \varepsilon \text{ and } f_R(r_1, r_2) = 0 \quad \Rightarrow \quad \varepsilon\varphi_F(\varepsilon)0 = 0 \text{ with } 0 \in \mathcal{IE}$$

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$

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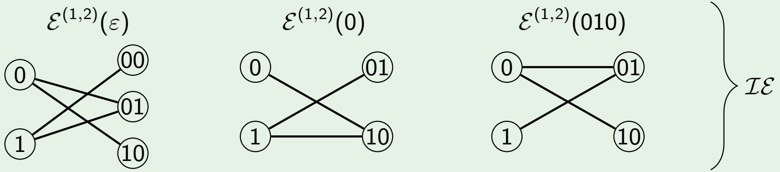

$\mathcal{BE}$:

- 0: $\forall$ bispecial pair $((\ell_1, r_1), (\ell_2, r_2)) \in B^{(1,2)}(0) \times B^{(1,2)}(0)$,

$$f_L(\ell_1, \ell_2) = \varepsilon \text{ and}$$

$$f_R(r_1, r_2) = 0 \qquad \Rightarrow \qquad \varepsilon\varphi_F(0)0 = 010 \text{ with } 010 \in \mathcal{IE}$$

$$\varphi_F : \begin{array}{ccc} 0 & \mapsto & 01, \\ 1 & \mapsto & 0. \end{array}$$

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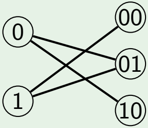
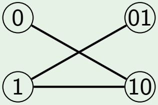
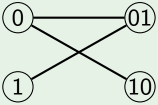
$\mathcal{E}^{(1,2)}(\varepsilon)$



$\mathcal{E}^{(1,2)}(0)$



$\mathcal{E}^{(1,2)}(010)$



$\mathcal{IE}$

$\mathcal{BE}$:

- 010: $\forall$ bispecial pair $((\ell_1, r_1), (\ell_2, r_2)) \in B^{(1,2)}(010) \times B^{(1,2)}(010)$, $f_L(\ell_1, \ell_2) = f_L(0, 1) = \varepsilon$ and $f_R(r_1, r_2) = f_R(01, 10) = 0$

$$\Rightarrow \varepsilon\varphi_F(010)0 = 010010 \text{ with } u = 010010 \notin \mathcal{IE}$$

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$

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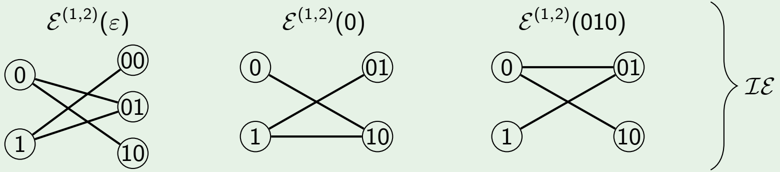
$\mathcal{IE}$**:** Bispecial words with length less than 4, since $4 = 2L + ||\varphi|| = 4$.



$\mathcal{BE}$:

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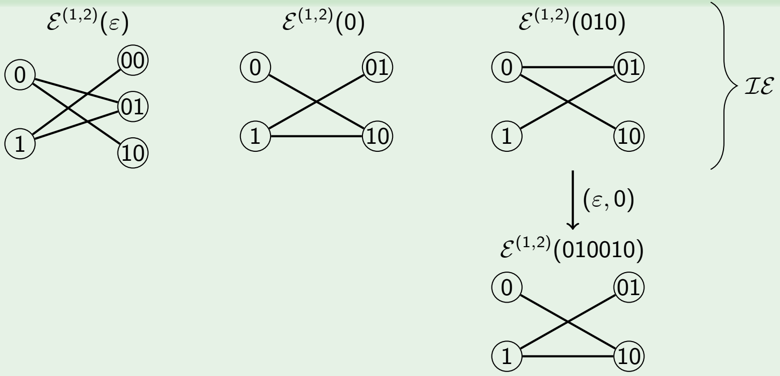
$$\Rightarrow \varepsilon\varphi_F(010)0 = 010010 \text{ with } u = 010010 \notin \mathcal{IE}$$

We should verify that the word 010 is the antecedent of u.

$$u = |01|0|01|0 = \varepsilon\varphi_F(010)0 \Rightarrow 010 \text{ is the antecedent of } u$$

$$\varphi_F : \begin{array}{rcl} 0 & \mapsto & 01, \\ 1 & \mapsto & 0. \end{array}$$

Example (Fibonacci)



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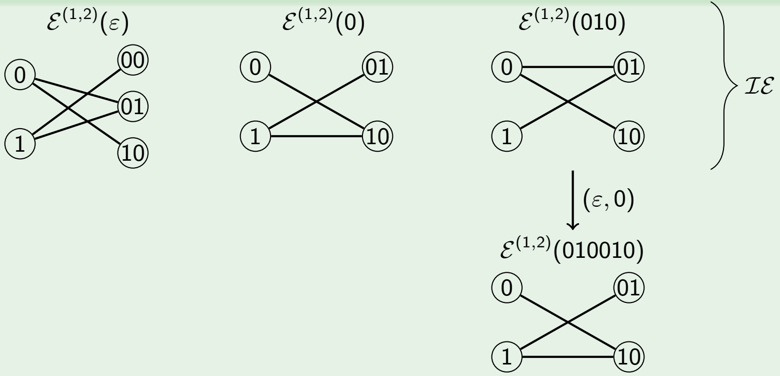

- $u = 010010$: $\forall$ bispecial pair $((\ell_1, r_1), (\ell_2, r_2)) \in B^{(1,2)}(\varepsilon) \times B^{(1,2)}(\varepsilon)$,

$$f_L(\ell_1, \ell_2) = \varepsilon \text{ and } f_R(r_1, r_2) = 0 \Rightarrow \varepsilon\varphi_F(u)0 = 01001010010 = w$$

Example (Fibonacci)

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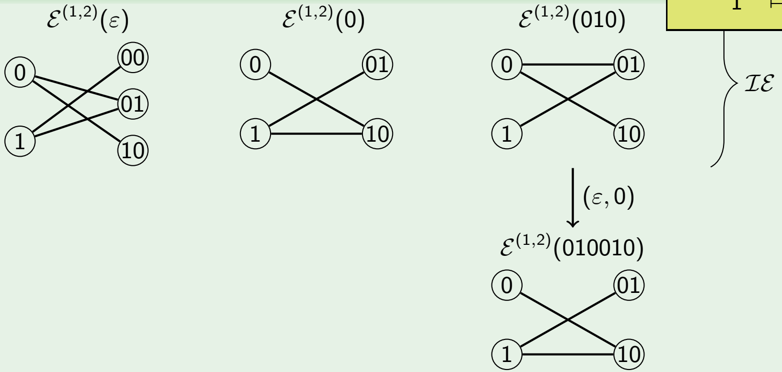

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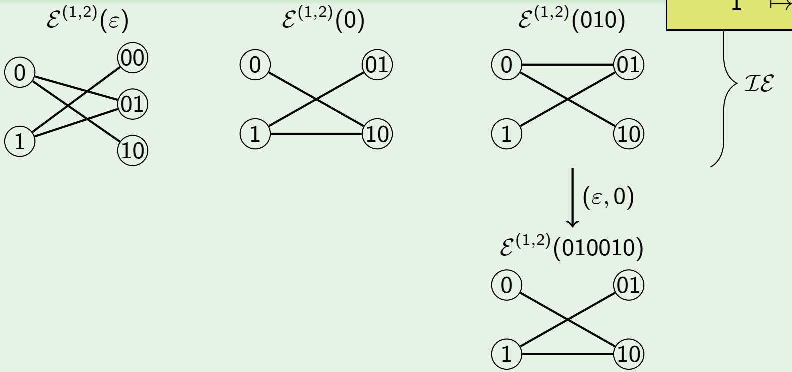
$01001010010 = |01|0|01|01|0|01|0 = \varepsilon\varphi_F(010010)0 \Rightarrow u$ is the antecedent of w

Example (Fibonacci)

$$\varphi_F : 0 \mapsto 01, \quad 1 \mapsto 0.$$



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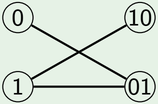

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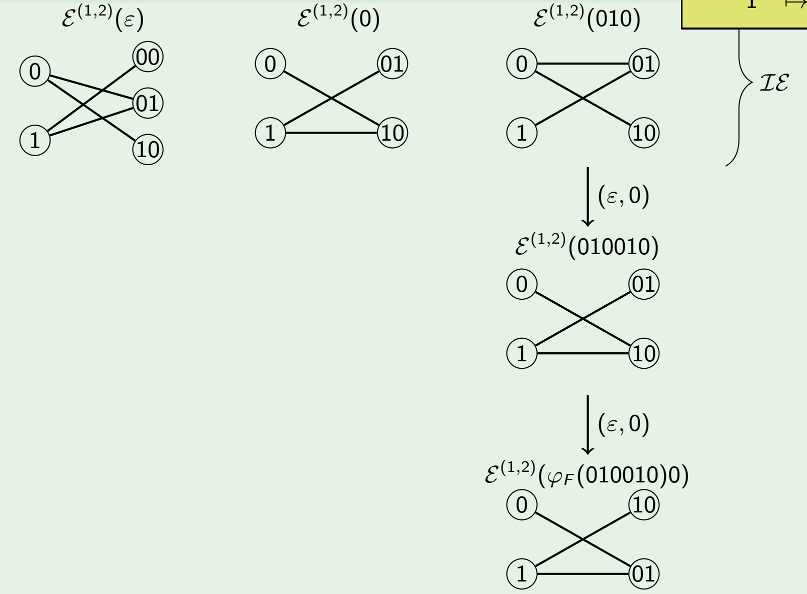

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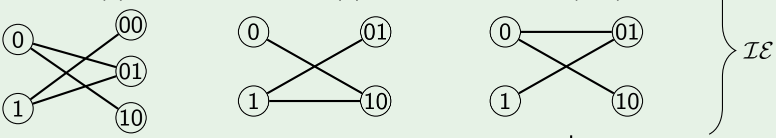

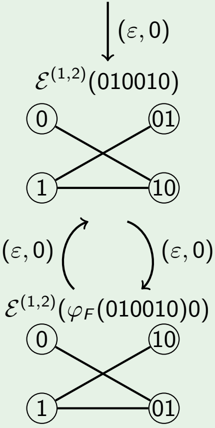

- $v = 01001010010$: $\varepsilon\varphi_F(v)0$

Proposition

The antecedent w of a bispecial word u, with $u = x\varphi(w)y$, is bispecial and also there is a bispecial pair $((\ell_1, r_1), (\ell_2, r_2)) \in B_{x,y}^{(C_L,C_R)}(w) \times B_{x,y}^{(C_L,C_R)}(w)$, with $x = f_L(\ell_1, \ell_2)$ and $y = f_R(r_1, r_2)$. In particular, $|y| < C_R$ and $|x| < C_L$.

$$C_L = \begin{cases} 1 & \text{if } \varphi \text{ is right-marked,} \\ ||\varphi|| & \text{if } \varphi \text{ is suffix, but not right-marked,} \\ L + ||\varphi|| & \text{otherwise;} \end{cases}$$

$$C_R = \begin{cases} 1 & \text{if } \varphi \text{ is left-marked,} \\ \min\{||\varphi||, L+1\} & \text{if } \varphi \text{ is prefix, but not left-marked,} \\ L+1 & \text{otherwise.} \end{cases}$$

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Let φ_F be the Fibonacci morphism with fixed point $\mathbf{z}$ the Fibonacci word, that is strongly recognizable for a constant $L = 1$.

- $\varphi_F(0) = 01 \in A^*1$ and $\varphi_F(1) = 0 \in A^*0 \Rightarrow \varphi_F$ is right-marked $\Rightarrow C_L = 1$.

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- $\varphi_F(0), \varphi_F(1) \in 0A^* \Rightarrow \varphi_F$ is not left-marked,
  $\varphi_F(0) = 01 = \varphi_F(1)1 \Rightarrow \varphi_F$ is not prefix,
  $\Rightarrow C_R = L + 1 = 2$.

Example (Fibonacci)

For the Fibonacci morphism φ_F: $C_L = 1$ and $C_R = 2$.

- $D_L = 1$

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$|\varphi(0)| = |01| = 2 \geq D_L + C_L - 1 = 1$
$|\varphi(1)| = |0| = 1 \geq D_L + C_L - 1 = 1$

- $D_R = 2$

$|\varphi(01)| = |010| = 3 \geq D_R + C_R - 1 = 3$
$|\varphi(00)| = |0101| = 4 \geq D_R + C_R - 1 = 3$
$|\varphi(10)| = |001| = 3 \geq D_R + C_R - 1 = 3$

Upper bound on the length of the words that we considered in order to construct the graph $\mathcal{K}$:

Proposition

Let φ be strongly two-sided recognizable morphism with the fixed point $\mathbf{z}$ for a constant $L > 0$. Let u be a bispecial factor of length n. Then all bispecial extended images of u have length at most $\Phi(n) = n||\varphi|| + C_L + C_R$.

Let $\delta = \delta(\mathcal{K}(\mathcal{L}(\mathbf{z})))$ be the length of the longest simple path in the graph $\mathcal{K}(\mathcal{L}(\mathbf{z}))$

Theorem

Let φ be a strongly two-sided recognizable morphism for a constant $L > 0$ on the adequate substitutive language $\mathcal{L}(\mathbf{z})$. For any long enough bispecial word $v \in \mathcal{L}(\mathbf{z})$ there is a bispecial word $w \in \mathcal{L}(\mathbf{z})$ with $|w| \leq \Phi^{\delta}(2L + ||\varphi|| - 1)$ such that $v \in [w]$.

Subshifts

Consider the morphism $\varphi : A^* \to A^*$.
The set,

$$X(\varphi) = \{z \in A^{\mathbb{Z}} | \mathcal{L}(z) \subset \mathcal{L}(\varphi)\}$$

is called the *subshift* of φ.

$$\begin{array}{rcl} \varphi : X(\varphi) & \to & X(\varphi) \\ z & \mapsto & \varphi(z) \end{array}$$

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Corollary [J. Almeida, A. Costa, R. Kyriakoglou, D. Perrin (2019)]

Let φ be a primitive aperiodic morphism, then the map $\varphi : X(\varphi) \to X(\varphi)$ is injective.